HISTORICAL ORIGIN OF GENETIC ALGEBRAS

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Abstract. *In this paper we give historical origin of genetic algebras, where mathematical methods have been applied successfully to population genetics for a long time. Investigation of mathematics to population genetics goes to Mendel's law, where he exploited symbols that are quite algebraically suggestive to express his genetic laws. Also non-Mendelian genetics is a basic language of molecular geneticists. Non-Mendelian genetics offers to mathematics new type of genetic algebras, which is called evolution algebras.*

Key words: *Algebra, genetic algebra, evolution algebra, genetics, Banach algebras, Mendelian and non-mendelian genetics;*

Аннотация. *В этой статье мы приводим исторические истоки генетических алгебр, где математические методы успешно применялись в популяционной генетике в течение длительного времени. Исследование математики в популяционной генетике восходит к закону Менделя, где он использовал символы, весьма алгебраически наводящие на размышления, чтобы выразить свои генетические законы. Кроме того, неменделевская генетика является основным языком молекулярных генетиков. Неменделевская генетика предлагает математике новый тип генетических алгебр, который называется эволюционной алгеброй.*

Ключевые слова: *алгебра, генетическая алгебра, эволюционная алгебра, генетика, банаховы алгебры, менделевская и неменделевская генетика;*

Annotatsiya. *Ushbu maqolada biz populyatsion genetikada uzoq vaqt davomida qo'llanib kelayotgan genetik algebralarning tarixiy kelib chiqishini taqdim etamiz. Populyatsion genetikada matematikani qo'lash Mendel qonuniga borib taqaladi, bunda olim genetik qonunlarini ifodalash uchun algebraik belgilardan foydalangan. Bundan tashqari, nomendel genetikasi molekulyar genetiklarning asosiy tilidir. Nomendel genetika matematikaga evolyutsion algebra deb ataladigan yangi turdagi genetik algebra tushunchasini olib kiradi.*

Kalit so'zlar: *algebra, genetik algebra, evolyutsion algebra, genetika, Banax algebralari, mendel va nomendel genetikasi;*

Historically, mathematical methods have been applied successfully to population genetics for a long time. Investigation of mathematics to population genetics goes to Mendel’s (Gregor Johann Mendel, 1822-1884) law, where he exploited symbols that are quite algebraically suggestive to express his genetic laws. Between 1856 and 1863, Mendel studied the hybridization of peas and gave the fundamental concept of the classical genetics, the gene, under the name “constant character” to explain the observed statistics of inheritance. In 1866, Gregor Mendel published the results of years of experimentation in breeding pea plants [17]. He showed that both parents must pass discrete physical factors which transmit information about their traits to their offspring at conception. Mendel first exploited symbols that are quite algebraically suggestive to express his genetic laws. Thus, mathematicians and geneticists once used nonassociative algebras to study Mendelian genetics and it was later termed “Mendelian algebras” by several other authors.

In the 1920s and 1930s as other major achievements of theoretical population genetics the Hardy-Weinberg law, the Fisher-Wright selection model, general genetic algebras were introduced. These are basic to many calculations in population genetics. The mathematics used in the classic works of Etherington, Fisher, Haldane and Wright was also not very complicated but was of great help for the theoretical understanding of evolutionary processes. At present, the methods of mathematical genetics are becoming more and more complex: the theory of probability is used, including the theory of random processes, non-linear differential and difference equations, and non-associative algebras.

Serebrowsky [28] was the first to give an algebraic interpretation of the sign x, which indicated sexual reproduction, and to give a mathematical formulation of Mendel’s laws in terms of non-associative algebras. Glivenkov [8] introduced the so-called Mendelian algebras for diploid populations with one locus or two unlinked loci. Independently, Kostitzin [13] also introduced a “symbolic multiplication” to express Mendel’s laws. The systematic study of algebras occurring in genetics can be attributed to series of papers of I.M.H. Etherington in 1939-1941. In [5-7] he succeeded in giving a precise mathematical formulation of Mendel’s laws in terms of nonassociative algebras, also covered many simple cases in which he considered algebras describing the genetics. Besides Etherington, fundamental contributions have been made by Abraham [1], Gonshor [9], Hench [10], Holgate [11,12], Reiser [25], Schafer [29]. Until 1980s, the most comprehensive reference in this area was Worz-Busekros's book [32], where author gave a rather complete presentation of algebras in genetics and applied the theory and the results to various concrete biological situations. More recent results, such as genetic evolution in genetic algebras, can be found in Lyubich's book [16]. A good survey is Reed's article [24].

Thus, Mendelian genetics introduced a new subject to mathematics: general genetic algebras. These algebras are in general commutative but not associative, furthermore they do not belong to any of the well-known classes of non-associative algebras such as Lie algebras, Jrdan algebras, or alternative algebras.

Baur [2] and Correns [4] first detected that chloroplast inheritance departed from Mendel's rules, and much later, mitochondrial gene inheritance was also identified in the same way, and non-Mendelian inheritance of organelle genes was recognized with two features — uniparental inheritance and vegetative segregation. Now, non-Mendelian genetics is a basic language of molecular geneticists. Non-Mendelian inheritance plays an important role in several disease processes. Non-Mendelian genetics offers to mathematics new type of genetic algebras, denominated evolution algebras, introduced in [30], these are algebras in which the multiplication tables are motivated by evolution laws of genetics. In [31] J.P. Tian established the foundation of the framework of evolution algebra theory and to discussed some applications of evolution algebras in non-Mendelian genetics, stochastic processes and genetics.

The systematic formulation of reproduction in non-Mendelian genetics as multiplication in algebras was introduced in [31] and called as "evolution algebras". These are algebras in which the multiplication tables are motivated by evolution laws of genetics.

Evolution algebras are defined in terms of generators and defining relations. Because the defining relations are unique for an evolution algebra, the generator set can serve as a basis for an evolution algebra. This property gives some advantage in studying evolution algebras.

Evolution algebras are not associative, in general; they are commutative, flexible, but not power-associative, in general; direct sums of evolution algebras are also evolution algebras; Kronecker products of evolutions algebras are also evolution algebras.

As a new type of algebra — the evolution algebras are non-associative and non-power-associative Banach algebras. Indeed, they are natural examples of non-associative complete normed algebras arising from science. Evolution algebras have connections with other fields of mathematics, including graph theory (particularly, random graphs and networks), group theory, Markov processes, dynamical systems.

Recently in [3] a notion of a chain of evolution algebras (CEA) is introduced. This chain is a dynamical system the state of which at each given time is an evolution algebra. The sequence of matrices of the structural constants for this chain of evolution algebras satisfies the Chapman-Kolmogorov equation.

In [23] a chain of n-dimensional evolution algebras corresponding to a permutation of n numbers defined. Also showed that a chain of evolution algebras (CEA) corresponding to a permutation is trivial (consisting only algebras with zero- multiplication) iff the permutation has not a fixed point.

Dynamics of two-dimensional chains of evolution algebras and their properties (baric, idempotent element and absolute nilpotent element transitions) are studied in [18-22,26]. Chain of evolution algebra of "chicken" population is studied in [27].

In [14] the notion of flow (depending on time) of finite-dimensional algebras introduced. A flow of algebras is a particular case of a continuous-time dynamical system whose states are finite-dimensional algebras with (cubic) matrices of structural constants satisfying an analogue of the Kolmogorov-Chapman equation. Since there are several kinds of multiplications between cubic matrices one has fix a multiplication first and then consider the Kolmogorov-Chapman equation with respect to the fixed multiplication. The existence of a solution for the Kolmogorov-Chapman equation provides the existence of flow of algebras. The time- dependent behavior of the different examples of flows constructed and studied in [15].

Before we discuss the mathematics of genetics, we need to acquaint ourselves with the necessary language from biology. The **genes** consist of the DNA (deoxyribonucleic acid), so called the chemical basis of heredity, which are carry hereditary information passed from parents to children. Genes are carried in the **chromosomes**, the physical basis of heredity, within the cell, which are package of **DNA** that carries genetic information. Each gene on a chromosome has different forms that it can take.

	A	a
A	AA	Aa
a	aA	aa

Table 1.1: Alleles passing from gametes to zygotes

These forms are called **alleles**, which produce variations in a genetically inherited trait. E.g., different alleles produce different hair colours - brown, blond, red, black, etc. Since humans are diploid organisms (meaning we have two sets of 23 chromosomes - one set from each parent), hair types are determined by two alleles. Haploid cells (or organisms) carry a single set of chromosomes. When diploid organisms reproduce, a process called meiosis produces gametes (sex cells) which carry a single set of chromosomes. When these gamete cells fuse (e.g., when sperm fertilizes egg), the result is a zygote, which is again a diploid cell, meaning it carries its hereditary information in a double set of chromosomes. When gametes fuse (or reproduce) to form zygotes a natural "multiplication" operation occurs.

Algebras arise in population genetic models in a quite natural way. If populations with one or several genetic traits are considered then each particular population is characterized by the frequencies of these traits, e.g. gene or genotype frequencies. These frequencies are collected into a vector. Convex combinations of such vectors describe mixtures of populations, thus the addition of vectors and the multiplication with constants have a meaning. Sexual reproduction leads to a multiplicative structure, as it is suggested by a formal representation of Mendel's laws.

Example 1. (Simple Mendelian Inheritance) [24]. As a natural first example, we consider simple Mendelian inheritance for a single gene with two alleles A and a. In this case, two gametes fusing (or reproducing) to form a zygote gives the multiplication table shown in Table 1, which in freshman biology class might be called a Punnett square.

The zygotes AA and aa are called homozygous, since they carry two copies of the same allele. In this case, simple Mendelian inheritance

	A	a
A	A	2 (A +
a	2 (*a* +	a

Table 1.2: Multiplication table of the gametic algebra for simple

Mendelian inheritance means that there is no chance involved as to what genetic information will be inherited in the next generation; i.e., AA will pass on the allele A and aa will pass on a. However, the zygotes Aa and aA (which are equivalent) each carry two different alleles. These zygotes are called heterozygous.

The rules of simple Mendelian inheritance indicate that the next generation will inherit either A or a with equal frequency. So, when two gametes reproduce, a multiplication is induced which indicates how the hereditary information will be passed down to the next generation. This multiplication is given by the following rules:

A x A = A (1.1)

A x a = -(A + a) (1.2)

a x A = -(a + A) (1.3)

ax a = a (1.4)

Rules (1.1) and (1.4) are expressions of the fact that if both gametes carry the same allele, then the offspring will inherit it. Rules (1.2) and (1.3) indicate that when gametes carrying A and a reproduce, half of the time the offspring will inherit A and the other half of the time it will inherit a. These rules are an algebraic representation of the rules of simple Mendelian inheritance. This multiplication table is shown in Table 1.2. We should point out that we are only concerning ourselves with genotypes (gene composition) and not phenotypes (gene expression). Hence we have made no mention of the dominant or recessive properties of our alleles.

Now we have defined a multiplication on the symbols A and *a* we can mathematically define the two-dimensional algebra over R with basis *A*, *a* and multiplication table as in Table 1.2. This algebra is called the ***gametic algebra*** for simple Mendelian inheritance with two alleles.

It is important to note that, there exist several classes of non-associative algebras (baric, evolution, Bernstein, train, stochastic, etc.), whose investigation has provided a number of significant contributions to theoretical population genetics. Such classes have been defined different times by several authors, and all algebras belonging to these classes are generally called "genetic"

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